Exercise 4.2

Unit 4 · Factorization and Algebraic Manipulation

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ $\quad a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

$(a + b)^3 = a^3 + b^3 + 3ab(a + b)$ $\quad (a - b)^3 = a^3 - b^3 - 3ab(a - b)$

Factorize the following.

1) $x^3 - 125$

$= (x)^3 - (5)^3$

$= (x - 5)[(x)^2 + (x)(5) + (5)^2]$

$\therefore (x - 5)(x^2 + 5x + 25)$

2) $8x^3 + 1$

$= (2x)^3 + (1)^3$

$= (2x + 1)[(2x)^2 - (2x)(1) + (1)^2]$

$\therefore (2x + 1)(4x^2 - 2x + 1)$

3) $3p^3q^3 - 81x^3$

$= 3(p^3q^3 - 27x^3)$

$= 3[(pq)^3 - (3x)^3]$

$= 3(pq - 3x)[(pq)^2 + (pq)(3x) + (3x)^2]$

$\therefore 3(pq - 3x)(p^2q^2 + 3pqx + 9x^2)$

4) $27 + 512x^3$

$= (3)^3 + (8x)^3$

$= (3 + 8x)[(3)^2 - (3)(8x) + (8x)^2]$

$\therefore (3 + 8x)(9 - 24x + 64x^2)$

5) $t^6 - 64$

$= (t^2)^3 - (4)^3$

$= (t^2 - 4)[(t^2)^2 + (t^2)(4) + (4)^2]$

$= (t^2 - 4)(t^4 + 4t^2 + 16)$

$= (t + 2)(t - 2)[(t^2)^2 + (4)^2 + 2(t^2)(4) - 2(t^2)(4) + 4t^2]$

$= (t + 2)(t - 2)[(t^2 + 4)^2 - 8t^2 + 4t^2]$

$= (t + 2)(t - 2)[(t^2 + 4)^2 - (2t)^2]$

$\therefore (t + 2)(t - 2)(t^2 - 2t + 4)(t^2 + 2t + 4)$

6) $x^6 + y^6$

$= (x^2)^3 + (y^2)^3$

$= (x^2 + y^2)[(x^2)^2 - (x^2)(y^2) + (y^2)^2]$

$\therefore (x^2 + y^2)(x^4 - x^2y^2 + y^4)$

7) $(2 - x)^3 + (y - 2)^3$

$= (y - x)[(2 - x)^2 - (2 - x)(y - 2) + (y - 2)^2]$

$= (y - x)[(4 - 4x + x^2) - (2y - 4 - xy + 2x)$

$+ (y^2 - 4y + 4)]$

$= (y - x)(x^2 + y^2 - 4x + 4 - 2y + 4 + xy$

$- 2x - 4y + 4)$

$\therefore (y - x)(x^2 + y^2 + xy - 6x - 6y + 12)$

Note: the term + xy is easy to miss in the last bracket; check it carefully.

8) $64(x + y)^3 - z^3$

$= [4(x + y)]^3 - (z)^3$

$= [4(x + y) - z][(4(x + y))^2 + 4(x + y)(z) + (z)^2]$

$= (4x + 4y - z)[16(x + y)^2 + 4z(x + y) + z^2]$

$= (4x + 4y - z)[16(x^2 + 2xy + y^2) + 4xz + 4yz + z^2]$

$\therefore (4x + 4y - z)(16x^2 + 16y^2 + 32xy + 4xz + 4yz + z^2)$

9) $27p^3 + 144pq^2 - 108p^2q - 64q^3$

$= 27p^3 - 64q^3 + 144pq^2 - 108p^2q$

$= (3p)^3 - (4q)^3 + 3(3p)(4q)^2 - 3(3p)^2(4q)$

$= (3p)^3 - (4q)^3 - 3(3p)(4q)(-4q + 3p)$

$= (3p)^3 - (4q)^3 - 3(3p)(4q)(3p - 4q)$

$\therefore (3p - 4q)^3$

10) $8p^3 + q^3 + 12p^2q + 6pq^2$

$= (2p)^3 + (q)^3 + 3(2p)^2(q) + 3(2p)(q)^2$

$= (2p)^3 + (q)^3 + 3(2p)(q)(2p + q)$

$\therefore (2p + q)^3$

11) $125x^3 - y^3 - 75x^2y + 15xy^2$

$= (5x)^3 - (y)^3 - 3(5x)^2(y) + 3(5x)(y)^2$

$= (5x)^3 - (y)^3 - 3(5x)(y)(5x - y)$

$\therefore (5x - y)^3$

12) $p^3 - 9p^2q + 27pq^2 - 27q^3$

$= (p)^3 - (3q)^3 - 3(p)^2(3q) + 3(p)(3q)^2$

$= (p)^3 - (3q)^3 - 3(p)(3q)(p - 3q)$

$\therefore (p - 3q)^3$

13) $(2x^2 - 3x + 6)(2x^2 - 3x) - 55$

Let $2x^2 - 3x = a$. Then the expression becomes

$(a + 6)(a) - 55 = a^2 + 6a - 55$

$= a^2 + 11a - 5a - 55 = a(a + 11) - 5(a + 11) = (a + 11)(a - 5)$

Back substitution: $(2x^2 - 3x + 11)(2x^2 - 3x - 5)$

$= (2x^2 - 3x + 11)(2x^2 - 5x + 2x - 5)$

$= (2x^2 - 3x + 11)[x(2x - 5) + 1(2x - 5)]$

$\therefore (2x^2 - 3x + 11)(2x - 5)(x + 1)$

14) $(y^2 + 2y - 3)(y^2 + 2y + 11) + 48$

Let $y^2 + 2y = a$. Then the expression becomes

$(a - 3)(a + 11) + 48 = a^2 - 3a + 11a - 33 + 48$

$= a^2 + 8a + 15 = a^2 + 3a + 5a + 15$

$= a(a + 3) + 5(a + 3) = (a + 3)(a + 5)$

Back substitution:

$\therefore (y^2 + 2y + 3)(y^2 + 2y + 5)$

15) $y(y - 1)(y - 3)(y - 4) + 2$

$= [y(y - 4)][(y - 1)(y - 3)] + 2$

$= (y^2 - 4y)(y^2 - y - 3y + 3) + 2$

$= (y^2 - 4y)(y^2 - 4y + 3) + 2$

Let $y^2 - 4y = a$. Then the expression becomes

$a(a + 3) + 2 = a^2 + 3a + 2 = a^2 + 2a + a + 2$

$= a(a + 2) + 1(a + 2) = (a + 2)(a + 1)$

Back substitution:

$\therefore (y^2 - 4y + 2)(y^2 - 4y + 1)$

16) $(k + 2)(k - 3)(k + 5)(k + 10) + 375$

$= [(k + 2)(k + 5)][(k - 3)(k + 10)] + 375$

$= (k^2 + 7k + 10)(k^2 + 7k - 30) + 375$

Let $k^2 + 7k = a$. Then the expression becomes

$(a + 10)(a - 30) + 375 = a^2 - 30a + 10a - 300 + 375$

$= a^2 - 20a + 75 = a^2 - 15a - 5a + 75$

$= a(a - 15) - 5(a - 15) = (a - 15)(a - 5)$

Back substitution:

$\therefore (k^2 + 7k - 15)(k^2 + 7k - 5)$

17) $(x-5)(x-6)(x+3)(x+2)+12$

$= [(x-5)(x+2)][(x-6)(x+3)] + 12$

$= (x^2-3x-10)(x^2-3x-18)+12$

Let $x^2-3x=a$. Then the expression becomes

$(a-10)(a-18)+12 = a^2-28a+180+12$

$= a^2-28a+192 = a^2-16a-12a+192$

$= a(a-16)-12(a-16) = (a-16)(a-12)$

Back substitution:

$\therefore (x^2-3x-16)(x^2-3x-12)$

18) $(x+1)(x+2)(x-3)(x-6)-21x^2$

$= [(x+1)(x-6)][(x+2)(x-3)] - 21x^2$

$= (x^2-5x-6)(x^2-x-6)-21x^2$

$= (x^2-6-5x)(x^2-6-x)-21x^2$

Let $x^2-6=y$. Then the expression becomes

$(y-5x)(y-x)-21x^2 = y^2-6xy+5x^2-21x^2$

$= y^2-6xy-16x^2 = y^2-8xy+2xy-16x^2$

$= y(y-8x)+2x(y-8x) = (y-8x)(y+2x)$

Replacing y by x^2-6:

$\therefore (x^2-8x-6)(x^2+2x-6)$

19) $(x-2)(x-6)(x-3)(x-4)-2x^2$

$= (x^2-8x+12)(x^2-7x+12)-2x^2$

$= (x^2+12-8x)(x^2+12-7x)-2x^2$

Let $x^2+12=y$. Then the expression becomes

$(y-8x)(y-7x)-2x^2 = y^2-15xy+56x^2-2x^2$

$= y^2-15xy+54x^2 = y^2-9xy-6xy+54x^2$

$= y(y-9x)-6x(y-9x) = (y-9x)(y-6x)$

Putting the value of y:

$\therefore (x^2-9x+12)(x^2-6x+12)$

20) $(5 - x)(2 + x)(10 - x)(1 + x) - 7x^2$

$= (10 + 3x - x^2)(10 + 9x - x^2) - 7x^2$

$= (10 - x^2 + 3x)(10 - x^2 + 9x) - 7x^2$

Let $10 - x^2 = y$. Then the expression becomes

$(y + 3x)(y + 9x) - 7x^2 = y^2 + 12xy + 27x^2 - 7x^2$

$= y^2 + 12xy + 20x^2 = y^2 + 10xy + 2xy + 20x^2$

$= y(y + 10x) + 2x(y + 10x) = (y + 10x)(y + 2x)$

Substituting y:

$\therefore (10 + 10x - x^2)(10 + 2x - x^2)$

21 **The expression $a^6 + 729$ can be written in two ways: (a) as a sum of two squares, (b) as a sum of two cubes.**

(a) $a^6 + 729 = (a^3)^2 + (27)^2$

A sum of two squares cannot be factorized (there is no formula for $a^2 + b^2$), so this form is not used.

(b) $a^6 + 729 = (a^2)^3 + (9)^3$

$= (a^2 + 9)[(a^2)^2 - (a^2)(9) + (9)^2]$

$\therefore (a^2 + 9)(a^4 - 9a^2 + 81)$

22 **Express $8 + 12t + 6t^2 + t^3$ as the product of three factors. Is each factor a binomial or a trinomial?**

$8 + 12t + 6t^2 + t^3 = t^3 + 8 + 6t^2 + 12t$

$= (t)^3 + (2)^3 + 3(t)^2(2) + 3(t)(2)^2$

$= (t)^3 + (2)^3 + 3(t)(2)(t + 2) = (t + 2)^3$

$\therefore (t + 2)(t + 2)(t + 2)$

Each factor, $t + 2$, has two terms, so each factor is a **binomial**.