# Miscellaneous Exercise 4

Unit 4 · Factorization and Algebraic Manipulation

**1 Encircle the correct option.**

**(i)** Factors of $-2 - a + a^2$ are

$\therefore$ (d) $(a + 1)(a - 2)$

$a^2 - a - 2 = a^2 - 2a + a - 2 = (a - 2)(a + 1)$

**(ii)** Factorization of $x^2 - x + \frac{1}{4}$ is

$\therefore$ (d) $\left(x - \frac{1}{2}\right)\left(x - \frac{1}{2}\right)$

It is a perfect square: $\left(x - \frac{1}{2}\right)^2$

**(iii)** $(x^2 + m^2)(x + m)(x^4 + m^4)(x - m)$ is the factored form of

$\therefore$ (c) $x^8 - m^8$

$(x - m)(x + m) = x^2 - m^2$; $\times (x^2 + m^2) = x^4 - m^4$; $\times (x^4 + m^4) = x^8 - m^8$

**(iv)** $a^4 + 64b^4$ is the product of

$\therefore$ (a) $a^2 - 4ab + 8b^2$ and $a^2 + 4ab + 8b^2$

$a^4 + 64b^4 = (a^2 + 8b^2)^2 - 16a^2b^2 = (a^2 + 8b^2)^2 - (4ab)^2$

**(v)** $x(2a + 3b + 7) + x(2a + 3b + 5)$ equals

$\therefore$ (a) $2x(2a + 3b + 6)$

$= x(4a + 6b + 12) = 2x(2a + 3b + 6)$

**(vi)** Which is the HCF of $-12x^2y^2$, $6xy^3$, $24x^2y^2$?

$\therefore$ (c) $6xy^2$

HCF of 12, 6, 24 is 6; lowest powers: $x$ and $y^2$

**(vii)** What is the LCM of $12x^2y^2$, $6xy^3$, $24x^2y^2$?

$\therefore$ (d) $24y^3x^2$

LCM of 12, 6, 24 is 24; highest powers: $x^2$ and $y^3$

**(viii)** What is the HCF of $7x - 6xy$ and $5xy^3 - 3x^2$?

$\therefore$ (c) $x$

$7x - 6xy = x(7 - 6y)$, $5xy^3 - 3x^2 = x(5y^3 - 3x)$

**(ix)** LCM of $7x - 6xy$ and $5y^3x - 3x^2$ is

$\therefore$ (d) $x(7 - 6y)(5y^3 - 3x)$

Common factor $x$ taken once, times the other factors.

**(x)** HCF of $7x^3 - 8y^3$ and $3x^3 - 5y^3$ is

$\therefore$ (a) $1$

The two expressions have no common factor.

**(xi)** LCM of $7x^3 - 8y^3$ and $3x^3 - 5y^3$ is

$\therefore$ (b) $(7x^3 - 8y^3)(3x^3 - 5y^3)$

With HCF = 1, the LCM is the product.

**(xii)** What is the product of $\frac{uv^2}{3w^3}$ and $\frac{6w^4}{u^2v^3}$?

$\therefore$ (b) $\frac{2w}{uv}$

$\frac{6uv^2w^4}{3u^2v^3w^3} = \frac{2w}{uv}$

**(xiii)** What is the quotient $\frac{3y^2}{10} \div \frac{y^3}{2}$?

$\therefore$ (b) $\frac{3}{5y}$

$= \frac{3y^2}{10} \times \frac{2}{y^3} = \frac{6}{10y} = \frac{3}{5y}$

**(xiv)** What is the sum of $\frac{2a}{a^2 - 1}$ and $\frac{-a}{a^2 - 1}$?

$\therefore$ (b) $\frac{a}{a^2 - 1}$

Same denominator: $\frac{2a - a}{a^2 - 1} = \frac{a}{a^2 - 1}$

**(xv)** What is the difference of $\frac{-3x}{x+y}$ and $\frac{x}{x+y}$?

$\therefore$ (d) $\frac{-4x}{x+y}$

$$\frac{-3x - x}{x+y} = \frac{-4x}{x+y}$$

**(xvi)** Product of two polynomials is $(a-b)^2(a^2+ab+b^2)$ and HCF is $a-b$. What is the LCM?

$\therefore$ (a) $a^3 - b^3$

$$LCM = \frac{product}{HCF} = (a-b)(a^2+ab+b^2) = a^3 - b^3$$

**(xvii)** Product of HCF and LCM is $(x^3 - y^3)(x+y)$. What is the product of the polynomials?

$\therefore$ (c) $(x^2 - y^2)(x^2 + xy + y^2)$

Product = HCF × LCM = $(x-y)(x^2+xy+y^2)(x+y) = (x^2-y^2)(x^2+xy+y^2)$

**(xviii)** What is the square root of $36x^6y^{16}$?

$\therefore$ (c) $6x^3y^8$

$\sqrt{36} = 6$, halve each power: $x^3$, $y^8$

**(xix)** What is the square root of $(15x^2 - 7y^2)^4$?

$\therefore$ (a) $\pm(15x^2 - 7y^2)^2$

Halve the power 4 to get 2; a square root has ± signs.

**2 Factorize the following.**

**$(a^2-5)^2 - 13(a^2-5) + 36$**

Let $x = a^2 - 5$. Then the expression becomes

$x^2 - 13x + 36 = x^2 - 9x - 4x + 36 = x(x-9) - 4(x-9)$

$= (x-4)(x-9)$

Back substitution: $(a^2 - 5 - 4)(a^2 - 5 - 9)$

$\therefore (a^2-9)(a^2-14) = (a+3)(a-3)(a^2-14)$

**3)** $6x^2 + 19x + 15$

$= 6x^2 + 9x + 10x + 15 = 3x(2x + 3) + 5(2x + 3)$

$\therefore (2x + 3)(3x + 5)$

**4)** $(x + 1)(x - 3)(x - 5)(x - 9) + 44$

$= [(x + 1)(x - 9)][(x - 3)(x - 5)] + 44$

$= (x^2 - 8x - 9)(x^2 - 8x + 15) + 44$

Let $x^2 - 8x = y$: $(y - 9)(y + 15) + 44$

$= y^2 - 9y + 15y - 135 + 44$

$= y^2 + 6y - 91 = y^2 + 13y - 7y - 91$

$= y(y + 13) - 7(y + 13) = (y + 13)(y - 7)$

$\therefore (x^2 - 8x + 13)(x^2 - 8x - 7)$

**5)** $x^4 - 12x^2 + 16$

$= (x^2)^2 + (4)^2 - 2(x^2)(4) + 2(x^2)(4) - 12x^2$

$= (x^2 - 4)^2 + 8x^2 - 12x^2 = (x^2 - 4)^2 - 4x^2$

$= (x^2 - 4)^2 - (2x)^2$

$\therefore (x^2 - 4 + 2x)(x^2 - 4 - 2x)$

**6)** $(3x^2 + 4x - 5)(3x^2 - 2 + 4x) - 4$

$= (3x^2 + 4x - 5)(3x^2 + 4x - 2) - 4$

Let $3x^2 + 4x = y$: $(y - 5)(y - 2) - 4$

$= y^2 - 5y - 2y + 10 - 4$

$= y^2 - 7y + 6 = y^2 - 6y - y + 6 = y(y - 6) - 1(y - 6)$

$= (y - 6)(y - 1)$

Back substitution:

$\therefore (3x^2 + 4x - 6)(3x^2 + 4x - 1)$

Note: $y - 1$ gives the second factor $3x^2 + 4x - 1$ (not $- 2$).

**7)** $2m^4 + m^2n^2 - 3n^4$

$= 2m^4 + 3m^2n^2 - 2m^2n^2 - 3n^4$

$= m^2(2m^2 + 3n^2) - n^2(2m^2 + 3n^2)$

$= (2m^2 + 3n^2)(m^2 - n^2)$

$\therefore (m + n)(m - n)(2m^2 + 3n^2)$

**8)** $x^4 + 10x^2y^2 - 56y^4$

$= x^4 + 14x^2y^2 - 4x^2y^2 - 56y^4$

$= x^2(x^2 + 14y^2) - 4y^2(x^2 + 14y^2)$

$= (x^2 + 14y^2)(x^2 - 4y^2)$

$\therefore (x + 2y)(x - 2y)(x^2 + 14y^2)$

**9)** $x^2 + 2ax - bx - 2ab$

$= x(x + 2a) - b(x + 2a)$

$\therefore (x + 2a)(x - b)$

**10)** $a^{12} - b^{12}$

$= (a^6)^2 - (b^6)^2 = (a^6 + b^6)(a^6 - b^6)$

$= (a^6 + b^6)[(a^3)^2 - (b^3)^2]$

$= (a^3 - b^3)(a^3 + b^3)(a^6 + b^6)$

$\therefore (a - b)(a^2 + ab + b^2)(a^3 + b^3)(a^6 + b^6)$

**11)** $28x^4y + 64x^3y - 60x^2y$

$= 4x^2y(7x^2 + 16x - 15) = 4x^2y(7x^2 + 21x - 5x - 15)$

$= 4x^2y[7x(x + 3) - 5(x + 3)]$

$\therefore 4x^2y(x + 3)(7x - 5)$

**12)** $4(x - y)^3 - (x - y)$

$= (x - y)[4(x - y)^2 - 1] = (x - y)[(2(x - y))^2 - (1)^2]$

$= (x - y)[2(x - y) - 1][2(x - y) + 1]$

$\therefore (x - y)(2x - 2y - 1)(2x - 2y + 1)$

**13)** $x^3p^2 - 8y^3p^2 - 4x^3q^2 + 32y^3q^2$

$= p^2(x^3 - 8y^3) - 4q^2(x^3 - 8y^3)$

$= (p^2 - 4q^2)[(x)^3 - (2y)^3]$

$\therefore (p + 2q)(p - 2q)(x - 2y)(x^2 + 2xy + 4y^2)$

**14)** $x^4 + y^4 - 7x^2y^2$

$= (x^2)^2 + (y^2)^2 + 2(x^2)(y^2) - 2(x^2)(y^2) - 7x^2y^2$

$= (x^2 + y^2)^2 - 9x^2y^2 = (x^2 + y^2)^2 - (3xy)^2$

$\therefore (x^2 + y^2 - 3xy)(x^2 + y^2 + 3xy)$

**15** **Find the highest common factor.**

**(i)** $x^3 + 3x^2 - 8x - 24,\ x^3 + 3x^2 - 3x - 9$

$x^3 + 3x^2 - 8x - 24 = x^2(x + 3) - 8(x + 3)$

$= (x + 3)(x^2 - 8)$

$x^3 + 3x^2 - 3x - 9 = x^2(x + 3) - 3(x + 3)$

$= (x + 3)(x^2 - 3)$

Common factor $= x + 3$

$\therefore$ Required HCF $= x + 3$

**(ii)** $3x^4 - 3x^3 - 2x^2 - x - 1,\ 9x^4 - 3x^3 - x - 1$

$$
\begin{array}{r|l}
 & 3 \\
3x^4 - 3x^3 - 2x^2 - x - 1 & 9x^4 - 3x^3 \quad - x \quad - 1 \\
 & -(9x^4 - 9x^3 - 6x^2 - 3x - 3) \\
\hline
 & \mathbf{6x^3 + 6x^2 + 2x + 2}
\end{array}
$$

$= 2(3x^3 + 3x^2 + x + 1)$

Leave out 2 and divide by $3x^3 + 3x^2 + x + 1$:

$$
\begin{array}{r|l}
 & x - 2 \\
3x^3 + 3x^2 + x + 1 & 3x^4 - 3x^3 - 2x^2 - x - 1 \\
 & -(3x^4 + 3x^3 + x^2 + x) \\
\hline
 & -6x^3 - 3x^2 - 2x - 1 \\
 & -(-6x^3 - 6x^2 - 2x - 2) \\
\hline
 & \mathbf{3x^2 + 1}
\end{array}
$$

Now divide $3x^3 + 3x^2 + x + 1$ by the remainder $3x^2 + 1$:

$$
\begin{array}{r|l}
 & x + 1 \\
3x^2 + 1 & 3x^3 + 3x^2 + x + 1 \\
 & -(3x^3 \quad + x) \\
\hline
 & 3x^2 + 1 \\
 & -(3x^2 + 1) \\
\hline
 & \mathbf{0}
\end{array}
$$

$\therefore$ Required HCF $= 3x^2 + 1$

**16** **Find the LCM.**

**(i)** $2x^2 + 3x + 1,\ 2x^2 + 5x + 2,\ x^2 + 3x + 2$

$2x^2 + 3x + 1 = 2x^2 + 2x + x + 1 = 2x(x + 1) + 1(x + 1)$

$= (x + 1)(2x + 1)$

$2x^2 + 5x + 2 = 2x^2 + 4x + x + 2 = 2x(x + 2) + 1(x + 2)$

$= (x + 2)(2x + 1)$

$x^2 + 3x + 2 = x^2 + 2x + x + 2 = x(x + 2) + 1(x + 2)$

$= (x + 2)(x + 1)$

Factors with highest powers: $x + 1,\ 2x + 1,\ x + 2$

$\therefore$ Required LCM $= (x + 1)(2x + 1)(x + 2)$

**(ii)** $3x^2 + 11x + 6,\ 3x^2 + 8x + 4,\ x^2 + 5x + 6$

$3x^2 + 11x + 6 = 3x^2 + 9x + 2x + 6 = 3x(x + 3) + 2(x + 3)$

$= (x + 3)(3x + 2)$

$3x^2 + 8x + 4 = 3x^2 + 6x + 2x + 4 = 3x(x + 2) + 2(x + 2)$

$= (x + 2)(3x + 2)$

$x^2 + 5x + 6 = x^2 + 3x + 2x + 6 = x(x + 3) + 2(x + 3)$

$= (x + 3)(x + 2)$

Factors with highest powers: $x + 3,\ 3x + 2,\ x + 2$

$\therefore$ Required LCM $= (x + 3)(3x + 2)(x + 2)$

**17** **Find the HCF and LCM of $a(a + c) - b(b + c),\ b(b + a) - c(c + a),\ c(c + b) - a(a + b)$.**

$a(a + c) - b(b + c) = a^2 - b^2 + ac - bc$

$= (a + b)(a - b) + c(a - b) = (a - b)(a + b + c)$

$b(b + a) - c(c + a) = b^2 - c^2 + ab - ac$

$= (b + c)(b - c) + a(b - c) = (b - c)(a + b + c)$

$c(c + b) - a(a + b) = c^2 - a^2 + bc - ab$

$= (c + a)(c - a) + b(c - a) = (c - a)(a + b + c)$

$\therefore$ HCF $= a + b + c$

$\therefore$ LCM $= (a - b)(b - c)(c - a)(a + b + c)$

**(18) Find the square root of**

$$\frac{9a^2}{x^2} - \frac{6a}{5x} + \frac{101}{25} - \frac{4x}{15a} + \frac{4x^2}{9a^2}$$

$$\frac{3a}{x} - \frac{1}{5} + \frac{2x}{3a}$$

| | |
|---|---|
| $\frac{3a}{x}$ | $\frac{9a^2}{x^2} - \frac{6a}{5x} + \frac{101}{25} - \frac{4x}{15a} + \frac{4x^2}{9a^2}$ |
| | $-\frac{9a^2}{x^2}$ |
| $\frac{6a}{x} - \frac{1}{5}$ | $-\frac{6a}{5x} + \frac{101}{25} - \frac{4x}{15a} + \frac{4x^2}{9a^2}$ |
| | $-(-\frac{6a}{5x} + \frac{1}{25})$ |
| $\frac{6a}{x} - \frac{2}{5} + \frac{2x}{3a}$ | $4 - \frac{4x}{15a} + \frac{4x^2}{9a^2}$ |
| | $-(4 - \frac{4x}{15a} + \frac{4x^2}{9a^2})$ |
| | $0$ |

$\therefore$ Square root $= \pm(\frac{3a}{x} - \frac{1}{5} + \frac{2x}{3a})$

**(19) Simplify.**

**(i)** $\frac{x^2 + x - 2}{x^2 - x - 20} \times \frac{x^2 + 5x + 4}{x^2 - x} \div \left(\frac{x^2 + 3x + 2}{x^2 - 2x - 15} \times \frac{x + 3}{x^2}\right)$

$$= \frac{(x+2)(x-1)}{(x-5)(x+4)} \times \frac{(x+4)(x+1)}{x(x-1)} \div \left(\frac{(x+2)(x+1)}{(x-5)(x+3)} \times \frac{x+3}{x^2}\right)$$

$$= \frac{(x+2)(x+1)}{(x-5)(x)} \times \frac{x^2(x-5)}{(x+2)(x+1)}$$

$\therefore x$

(ii) $\dfrac{1}{x+1} - \dfrac{1}{(x+1)(x+2)} + \dfrac{1}{(x+1)(x+2)(x+3)}$

$= \dfrac{(x+2)(x+3) - (x+3) + 1}{(x+1)(x+2)(x+3)} = \dfrac{x^2 + 5x + 6 - x - 3 + 1}{(x+1)(x+2)(x+3)}$

$= \dfrac{x^2 + 4x + 4}{(x+1)(x+2)(x+3)} = \dfrac{(x+2)^2}{(x+1)(x+2)(x+3)}$

$\therefore \dfrac{x+2}{(x+1)(x+3)}$

**20** **Find a and b if $x^4 + ax^3 + bx^2 - 4x + 4$ is a perfect square.**

**Method 1** (descending powers of x)

$x^2 - x + 2$

| | |
|---|---|
| $x^2$ | $x^4 + ax^3 + bx^2 - 4x + 4$ |
| | $-x^4$ |
| $2x^2 - x$ | $ax^3 + bx^2 - 4x + 4$ |
| | $-(-2x^3 + x^2)$ |
| $2x^2 - 2x + 2$ | $(a+2)x^3 + (b-1)x^2 - 4x + 4$ |
| | $-(4x^2 - 4x + 4)$ |
| | $(a+2)x^3 + (b-5)x^2$ |

The expression is a perfect square only if the remainder is zero:

$a + 2 = 0 \Rightarrow a = -2$ and $b - 5 = 0 \Rightarrow b = 5$

$\therefore a = -2,\ b = 5$

Check: $(x^2 - x + 2)^2 = x^4 - 2x^3 + 5x^2 - 4x + 4$.

**Method 2** (ascending powers of x)

$$\begin{array}{r|l}
 & \mathbf{2 - x + x^2} \\
2 & 4 - 4x + bx^2 + ax^3 + x^4 \\
 & -4 \\
\hline
4 - x & -4x + bx^2 + ax^3 + x^4 \\
 & -(-4x + x^2) \\
\hline
4 - 2x + x^2 & (b - 1)x^2 + ax^3 + x^4 \\
 & -(4x^2 - 2x^3 + x^4) \\
\hline
 & (b - 5)x^2 + (a + 2)x^3
\end{array}$$

As above, $a = -2$ and $b = 5$.