Exercise 6.4

Unit 6 · Trigonometry and Bearing

$\sin^2\theta + \cos^2\theta = 1 \quad 1 + \tan^2\theta = \sec^2\theta \quad 1 + \cot^2\theta = \operatorname{cosec}^2\theta$

1 Prove the following.

(i) $(1 - \sin^2\theta)\sec^2\theta = 1$

$$L.H.S = (1 - \sin^2\theta)\sec^2\theta = (\cos^2\theta)\left(\frac{1}{\cos^2\theta}\right)$$

since $1 - \sin^2\theta = \cos^2\theta$

$= 1 = R.H.S$

$\therefore$ Proved

(ii) $\dfrac{1 - \sin\theta}{1 + \sin\theta} = (\sec\theta - \tan\theta)^2$

$$R.H.S = \left(\frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}\right)^2 = \frac{(1 - \sin\theta)^2}{\cos^2\theta} = \frac{(1 - \sin\theta)^2}{1 - \sin^2\theta}$$

$$= \frac{(1 - \sin\theta)^2}{(1 - \sin\theta)(1 + \sin\theta)} = \frac{1 - \sin\theta}{1 + \sin\theta} = L.H.S$$

$\therefore$ Proved

(iii) $\sqrt{\dfrac{1 - \cos\theta}{1 + \cos\theta}} = \dfrac{\sin\theta}{1 + \cos\theta}$

$$L.H.S = \sqrt{\frac{1 - \cos\theta}{1 + \cos\theta} \times \frac{1 + \cos\theta}{1 + \cos\theta}} = \sqrt{\frac{1 - \cos^2\theta}{(1 + \cos\theta)^2}}$$

$$= \sqrt{\frac{\sin^2\theta}{(1 + \cos\theta)^2}} = \frac{\sin\theta}{1 + \cos\theta} = R.H.S$$

$\therefore$ Proved

(iv) $\dfrac{\sin\theta - 2\sin^3\theta}{2\cos^3\theta - \cos\theta} = \tan\theta$

$$L.H.S = \frac{\sin\theta(1 - 2\sin^2\theta)}{\cos\theta(2\cos^2\theta - 1)} = \tan\theta \cdot \frac{1 - 2(1 - \cos^2\theta)}{2\cos^2\theta - 1}$$

$$= \tan\theta \cdot \frac{2\cos^2\theta - 1}{2\cos^2\theta - 1} = \tan\theta = R.H.S$$

$\therefore$ Proved

(v) $\sqrt{\sec^2\theta + \operatorname{cosec}^2\theta} = \tan\theta + \cot\theta$

$$L.H.S = \sqrt{1 + \tan^2\theta + 1 + \cot^2\theta} = \sqrt{2 + \tan^2\theta + \frac{1}{\tan^2\theta}}$$

$$= \sqrt{\frac{\tan^4\theta + 2\tan^2\theta + 1}{\tan^2\theta}} = \sqrt{\frac{(\tan^2\theta + 1)^2}{\tan^2\theta}}$$

$$= \frac{\tan^2\theta + 1}{\tan\theta} = \tan\theta + \frac{1}{\tan\theta} = \tan\theta + \cot\theta = R.H.S$$

$\therefore$ Proved

(vi) $\dfrac{1}{\sec\theta - \tan\theta} = \sec\theta + \tan\theta$

$$L.H.S = \frac{1}{\sec\theta - \tan\theta} \times \frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta} = \frac{\sec\theta + \tan\theta}{\sec^2\theta - \tan^2\theta}$$

$$= \frac{\sec\theta + \tan\theta}{1} = \sec\theta + \tan\theta = R.H.S$$

since $\sec^2\theta - \tan^2\theta = 1$

$\therefore$ Proved

(vii) $\sin^2\theta + \cos^4\theta = \sin^4\theta + \cos^2\theta$

$$L.H.S = \sin^2\theta + (\cos^2\theta)^2 = \sin^2\theta + (1 - \sin^2\theta)^2$$

$$= \sin^2\theta + 1 - 2\sin^2\theta + \sin^4\theta = (1 - \sin^2\theta) + \sin^4\theta$$

$$= \cos^2\theta + \sin^4\theta = R.H.S$$

$\therefore$ Proved

(viii) $\dfrac{\tan\theta + \sin\theta}{\tan\theta - \sin\theta} = \dfrac{\sec\theta + 1}{\sec\theta - 1}$

$$L.H.S = \frac{\frac{\sin\theta}{\cos\theta} + \sin\theta}{\frac{\sin\theta}{\cos\theta} - \sin\theta} = \frac{\sin\theta + \sin\theta\cos\theta}{\sin\theta - \sin\theta\cos\theta}$$

$$= \frac{\sin\theta(1 + \cos\theta)}{\sin\theta(1 - \cos\theta)} = \frac{1 + \cos\theta}{1 - \cos\theta} = \frac{1 + \frac{1}{\sec\theta}}{1 - \frac{1}{\sec\theta}}$$

$$= \frac{\sec\theta + 1}{\sec\theta - 1} = R.H.S$$

$\therefore$ Proved

(ix) $\sec^2\theta + \operatorname{cosec}^2\theta = \sec^2\theta \operatorname{cosec}^2\theta$

$$L.H.S = \frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta} = \frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta\cos^2\theta} = \frac{1}{\sin^2\theta\cos^2\theta}$$

$$= \frac{1}{\cos^2\theta} \cdot \frac{1}{\sin^2\theta} = \sec^2\theta \operatorname{cosec}^2\theta = R.H.S$$

$\therefore$ Proved

(x) $\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta\cos^2\theta$

$$L.H.S = (\sin^2\theta)^3 + (\cos^2\theta)^3$$

using $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$

$$= (\sin^2\theta + \cos^2\theta)(\sin^4\theta + \cos^4\theta - \sin^2\theta\cos^2\theta)$$

$$= (1)[(\sin^4\theta + \cos^4\theta + 2\sin^2\theta\cos^2\theta) - 3\sin^2\theta\cos^2\theta]$$

$$= (\sin^2\theta + \cos^2\theta)^2 - 3\sin^2\theta\cos^2\theta = 1 - 3\sin^2\theta\cos^2\theta = R.H.S$$

$\therefore$ Proved

2 If $x\cos\theta + y\sin\theta = m$ and $x\sin\theta - y\cos\theta = n$, prove $x^2 + y^2 = m^2 + n^2$.

Squaring and adding the two equations:

$(x\cos\theta + y\sin\theta)^2 + (x\sin\theta - y\cos\theta)^2 = m^2 + n^2$

$x^2\cos^2\theta + y^2\sin^2\theta + 2xy\sin\theta\cos\theta$

$\quad + x^2\sin^2\theta + y^2\cos^2\theta - 2xy\sin\theta\cos\theta = m^2 + n^2$

$x^2(\cos^2\theta + \sin^2\theta) + y^2(\sin^2\theta + \cos^2\theta) = m^2 + n^2$

$\therefore x^2 + y^2 = m^2 + n^2$

3 Solve for θ $(0 \le \theta \le \frac{\pi}{2})$.

(i) $2\sin^2\theta = \frac{1}{2}$

$\sin^2\theta = \frac{1}{4} \Rightarrow \sin\theta = \pm\frac{1}{2}$; in $0 \le \theta \le \frac{\pi}{2}$, $\sin\theta = \frac{1}{2}$

$\therefore \theta = 30°$

(ii) $\sin\theta = \cos\theta$

$\frac{\sin\theta}{\cos\theta} = 1 \Rightarrow \tan\theta = 1$

$\therefore \theta = 45°$

(iii) $\sec^2\theta - 2\tan^2\theta = 0$

$1 + \tan^2\theta - 2\tan^2\theta = 0 \Rightarrow \tan^2\theta = 1 \Rightarrow \tan\theta = \pm 1$

In $0 \le \theta \le \frac{\pi}{2}$, $\tan\theta = 1$

$\therefore \theta = 45°$

5 **If $x = r\cos\alpha\sin\beta$, $y = r\cos\alpha\cos\beta$ and $z = r\sin\alpha$, prove that $x^2 + y^2 + z^2 = r^2$.**

Squaring and adding:

$$x^2 + y^2 + z^2 = r^2\cos^2\alpha\sin^2\beta + r^2\cos^2\alpha\cos^2\beta + r^2\sin^2\alpha$$

$$= r^2[\cos^2\alpha(\sin^2\beta + \cos^2\beta) + \sin^2\alpha]$$

$$= r^2(\cos^2\alpha + \sin^2\alpha) = r^2(1)$$

$$\therefore x^2 + y^2 + z^2 = r^2$$