# Exercise 8.1

Unit 8 · Geometry of Straight Lines

Gradient (slope): $m = \tan\theta$ or $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Parallel lines: $m_1 = m_2$ Perpendicular lines: $m_1 \cdot m_2 = -1$

**1 Find the gradient (slope) of the line whose inclination is given.**

**(i)** $\theta = 0°$

$m = \tan\theta = \tan 0°$

$\therefore m = 0$

**(ii)** $\theta = 30°$

$m = \tan\theta = \tan 30°$

$\therefore m = 0.577$

**(iii)** $\theta = 60°$

$m = \tan\theta = \tan 60°$

$\therefore m = 1.732$

**(iv)** $\theta = 90°$

$m = \tan\theta = \tan 90°$

$\therefore m$ is undefined

**(v)** $\theta = 120°$

$m = \tan\theta = \tan 120°$

$\therefore m = -1.732$

**(vi)** $\theta = 150°$

$m = \tan\theta = \tan 150°$

$\therefore m = -0.577$

**(vii)** $\theta = 170°$

$m = \tan\theta = \tan 170°$

$\therefore m = -0.176$

**(viii)** $\theta = 45.5°$

$m = \tan\theta = \tan 45.5°$

$\therefore m = 1.018$

In (iv), tan 90° is not defined, so a vertical line has no gradient (its slope is undefined, sometimes written ∞).

**2 Find the inclination of the line whose gradient is given.**

**(i)** $m = 0$

$\tan\theta = 0$

$\theta = \tan^{-1}(0)$

$\therefore \theta = 0°$

**(ii)** $m = 0.577$

$\tan\theta = 0.577$

$\theta = \tan^{-1}(0.577)$

$\therefore \theta = 30°$

**(iii)** $m = -1.732$

$\theta = \tan^{-1}(-1.732) = -60°$

$\theta = 180° - 60°$

$\therefore \theta = 120°$

**(iv)** $m = -0.364$

$\theta = \tan^{-1}(-0.364) = -20°$

$\theta = 180° - 20°$

$\therefore \theta = 160°$

*Inclination is measured between 0° and 180°, so a negative angle from the calculator is changed by adding 180°.*

**3** **Find the gradient and the inclination of the line through each pair of points.**

**(i)** $A(2, 6)$ and $B(5, 8)$

Gradient of $AB = \dfrac{8 - 6}{5 - 2} = \dfrac{2}{3}$

Inclination $\theta = \tan^{-1}\left(\dfrac{2}{3}\right) = 33.69°$

$\therefore m = \dfrac{2}{3}, \quad \theta = 33.69°$

**(ii)** $C(-2, 4)$ and $D(1, -3)$

Gradient of $CD = \dfrac{-3 - 4}{1 - (-2)} = \dfrac{-7}{3}$

Inclination $\theta = \tan^{-1}\left(-\dfrac{7}{3}\right) = -66.80°$

$\Rightarrow \quad \theta = 180° - 66.80° = 113.20°$

$\therefore m = \dfrac{-7}{3}, \quad \theta = 113.20°$

**(iii)** $E(5, -2)$ and $F(-2, -3)$

$$\text{Gradient of } EF = \frac{-3-(-2)}{-2-5} = \frac{-1}{-7} = \frac{1}{7}$$

$$\text{Inclination } \theta = \tan^{-1}\left(\frac{1}{7}\right) = 8.13°$$

$$\therefore m = \frac{1}{7}, \quad \theta = 8.13°$$

**4** **Find the slope of a line (i) parallel to and (ii) perpendicular to the line through A(−2, 6) and B(7, −3).**

$$\text{Gradient of } AB = \frac{-3-6}{7-(-2)} = \frac{-9}{9} = -1$$

**(i)** Parallel lines have equal slopes: slope $= -1$

**(ii)** $m_1 \cdot m_2 = -1$

$$\Rightarrow m_2 = \frac{-1}{m_1} = \frac{-1}{-1} = 1$$

$\therefore$ Parallel slope $= -1$, perpendicular slope $= 1$

**5** **The slope of the line through A(3, x) and B(5, 8) is 4. Find x.**

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\Rightarrow 4 = \frac{8-x}{5-3} = \frac{8-x}{2}$$

$$8 - x = 8 \Rightarrow x = 0$$

$\therefore x = 0$

**6** **AB is parallel to CD, where A(k, 2), B(3, 5), C(5, −1) and D(8, 7). Find k.**

$$m_1 = \text{slope of } AB = \frac{5-2}{3-k} = \frac{3}{3-k}$$

$$m_2 = \text{slope of } CD = \frac{7-(-1)}{8-5} = \frac{8}{3}$$

$$AB \parallel CD \Rightarrow m_1 = m_2 \Rightarrow \frac{3}{3-k} = \frac{8}{3}$$

$$9 = 24 - 8k \Rightarrow 8k = 15$$

$$\therefore k = \frac{15}{8}$$

**7** **PQ is perpendicular to RS, where P(−1, 2), Q(4, 7), R(2, k) and S(7, 10). Find k.**

$$m_1 = \text{slope of } PQ = \frac{7-2}{4-(-1)} = \frac{5}{5} = 1$$

$$m_2 = \text{slope of } RS = \frac{10-k}{7-2} = \frac{10-k}{5}$$

$$PQ \perp RS \Rightarrow m_1 \cdot m_2 = -1$$

$$\Rightarrow (1)\left(\frac{10-k}{5}\right) = -1$$

$$10 - k = -5 \Rightarrow k = 15$$

$$\therefore k = 15$$

**8** **Show that the points X(0, −3), Y(4, 7) and Z(6, 12) are collinear.**

$$\text{Slope of } XY = \frac{7-(-3)}{4-0} = \frac{10}{4} = \frac{5}{2}$$

$$\text{Slope of } YZ = \frac{12-7}{6-4} = \frac{5}{2}$$

$$\text{Slope of } XZ = \frac{12-(-3)}{6-0} = \frac{15}{6} = \frac{5}{2}$$

The slopes of XY, YZ and XZ are equal.

$\therefore$ X, Y and Z are collinear.

**9** **The points P(4, y), Q(5, 2) and R(6, 2y + 1) are collinear. Find y.**

$$\text{Slope of } PQ = \frac{2-y}{5-4} = 2-y$$

$$\text{Slope of } QR = \frac{2y+1-2}{6-5} = 2y-1$$

Collinear points have equal slopes: $2 - y = 2y - 1$

$2 + 1 = 2y + y \Rightarrow 3 = 3y$

$\therefore y = 1$

Check: slope of $PR = \frac{(2y+1)-y}{6-4} = \frac{y+1}{2} = 1$ when $y = 1$.

**10** **Show that A(3, −1), B(−5, −5) and C(1, 3) are the vertices of a right-angled triangle.**

$$m_1 = \text{slope of } AB = \frac{-5-(-1)}{-5-3} = \frac{-4}{-8} = \frac{1}{2}$$

$$m_2 = \text{slope of } BC = \frac{3-(-5)}{1-(-5)} = \frac{8}{6} = \frac{4}{3}$$

$$m_3 = \text{slope of } CA = \frac{3-(-1)}{1-3} = \frac{4}{-2} = -2$$

$$m_1 \cdot m_3 = \frac{1}{2} \times (-2) = -1, \quad \text{so } AB \perp CA$$

The right angle is at A.

∴ ABC is a right-angled triangle.

**11** **Show that A(−2, 1), B(6, 3), C(10, 5) and D(2, 3) are the vertices of a parallelogram.**

$$m_1 = \text{slope of } AB = \frac{3-1}{6-(-2)} = \frac{2}{8} = \frac{1}{4}$$

$$m_3 = \text{slope of } CD = \frac{3-5}{2-10} = \frac{-2}{-8} = \frac{1}{4}$$

$$m_2 = \text{slope of } BC = \frac{5-3}{10-6} = \frac{2}{4} = \frac{1}{2}$$

$$m_4 = \text{slope of } AD = \frac{3-1}{2-(-2)} = \frac{2}{4} = \frac{1}{2}$$

$$m_1 = m_3 \Rightarrow AB \parallel CD, \quad m_2 = m_4 \Rightarrow BC \parallel AD$$

Also $m_1 \cdot m_2 \neq -1$, so the sides are not perpendicular.

∴ Both pairs of opposite sides are parallel, so ABCD is a parallelogram.

**(12) P(x, y), Q(−2, 2), R(1, 4) and S(10, 1) are the vertices of a parallelogram PQRS. Find x and y.**

$m_1$ = slope of $PQ = \dfrac{2-y}{-2-x}$

$m_2$ = slope of $QR = \dfrac{4-2}{1-(-2)} = \dfrac{2}{3}$

$m_3$ = slope of $RS = \dfrac{1-4}{10-1} = \dfrac{-3}{9} = -\dfrac{1}{3}$

$m_4$ = slope of $PS = \dfrac{1-y}{10-x}$

$PQ \parallel RS$: $\dfrac{2-y}{-2-x} = \dfrac{-1}{3} \Rightarrow 6 - 3y = 2 + x$

$\Rightarrow x + 3y = 4 \quad ...(i)$

$QR \parallel PS$: $\dfrac{2}{3} = \dfrac{1-y}{10-x} \Rightarrow 20 - 2x = 3 - 3y$

$\Rightarrow 2x - 3y = 17 \quad ...(ii)$

Adding (i) and (ii): $3x = 21 \Rightarrow x = 7$

Put $x = 7$ in (i): $7 + 3y = 4 \Rightarrow 3y = -3 \Rightarrow y = -1$

$\therefore x = 7, \; y = -1$, i.e. $P(7, -1)$

**13** **A(2, −1), B(3, 4) and C(−2, 3) are three vertices of a rhombus ABCD. Find the fourth vertex D.**

Let the fourth vertex be $D(x, y)$.

$$m_1 = \text{slope of } AB = \frac{4 - (-1)}{3 - 2} = 5$$

$$m_2 = \text{slope of } BC = \frac{3 - 4}{-2 - 3} = \frac{-1}{-5} = \frac{1}{5}$$

$$m_3 = \text{slope of } DC = \frac{y - 3}{x + 2}$$

$$m_4 = \text{slope of } AD = \frac{y + 1}{x - 2}$$

$$AB \parallel DC: \quad 5 = \frac{y - 3}{x + 2} \Rightarrow 5x + 10 = y - 3$$

$$\Rightarrow 5x - y = -13 \quad \ldots(i)$$

$$BC \parallel AD: \quad \frac{1}{5} = \frac{y + 1}{x - 2} \Rightarrow x - 2 = 5y + 5$$

$$\Rightarrow x = 7 + 5y \quad \ldots(ii)$$

Put (ii) in (i): $5(7 + 5y) - y = -13 \Rightarrow 35 + 24y = -13$

$\Rightarrow 24y = -48 \Rightarrow y = -2$

From (ii): $x = 7 + 5(-2) = -3$

$\therefore$ Fourth vertex $D(-3, -2)$

Check: $AB = BC = \sqrt{26}$, so ABCD is a rhombus.

**14** **A(5, 0), B(0, 5) and C(8, 8) are the vertices of a triangle. Find the slopes of (i) its sides, (ii) its medians and (iii) its altitudes.**

**(i)** Slopes of the sides

$$\text{Slope of } AB = \frac{5-0}{0-5} = -1$$

$$\text{Slope of } BC = \frac{8-5}{8-0} = \frac{3}{8}$$

$$\text{Slope of } AC = \frac{8-0}{8-5} = \frac{8}{3}$$

**(ii)** Slopes of the medians. Let D, E and F be the midpoints of AB, BC and AC.

$$D\left(\frac{5+0}{2}, \frac{0+5}{2}\right) = D\left(\frac{5}{2}, \frac{5}{2}\right)$$

$$E\left(\frac{0+8}{2}, \frac{5+8}{2}\right) = E\left(4, \frac{13}{2}\right)$$

$$F\left(\frac{5+8}{2}, \frac{0+8}{2}\right) = F\left(\frac{13}{2}, 4\right)$$

$$\text{Slope of median } CD = \frac{\frac{5}{2}-8}{\frac{5}{2}-8} = 1$$

$$\text{Slope of median } AE = \frac{\frac{13}{2}-0}{4-5} = -\frac{13}{2}$$

$$\text{Slope of median } BF = \frac{4-5}{\frac{13}{2}-0} = -\frac{2}{13}$$

**(iii)** Slopes of the altitudes (each altitude is perpendicular to a side, so its slope = $-1 \div$ slope of that side)

Altitude on AB: $\frac{-1}{-1} = 1$

Altitude on BC: $\frac{-1}{\frac{3}{8}} = -\frac{8}{3}$

Altitude on AC: $\frac{-1}{\frac{8}{3}} = -\frac{3}{8}$